7th European Workshop on Structural Health Monitoring
July 8-11, 2014. La Cité, Nantes, France

Detection of structural modifications of an Euler-Bernoulli beam through modal parameters variation

Gwendal Cumunel[1], **Thi-Thu-Ha Le**[1], **Nelly Point**[1], **Pierre Argoul**[1]

[1] *Université Paris-Est, Laboratoire Navier (ENPC / IFSTTAR / CNRS), Ecole des Ponts ParisTech, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne-la-Vallée Cedex 2, France*

gwendal.cumunel@enpc.fr

Abstract

This paper deals with a class of identification problems mainly concerning the localization of structural modifications in elastic beams with or without tension by dynamic analysis. Variations of tension, mass density and/or bending stiffness induce variations of the beam natural frequencies. For each natural frequency, a first order estimate of this modification is then established. It depends on the variations of tension, density and bending stiffness and also on the corresponding mode shape and curvature mode shape of the initial state. When the variations of density and flexural rigidity are localized along the beam (as for a notch in a beam or a crack in a cable), this estimate is used to propose a procedure permitting detection, localization and, eventually, quantification of these variations knowing the modifications of the first natural frequencies and the variation of tension. This procedure is applied successfully to experimental data found in literature concerning beams without tension and only bending stiffness modifications.

Keywords : *structural health monitoring, damage detection, localization of cracks, modal analysis, Euler-Bernoulli beam*

Introduction

The demand for enhanced performance and reliability of structures in terms of safety, noise and durability is ever increasing. A lot of research has been done for decades on structural health monitoring using a large number of different techniques (based on ultrasound [1], genetic algorithms [2], or vibration methods [3, 4] for example) and applied in various fields like aeronautics [5], industry [6] or civil engineering [7].

Beams are traditionally used to model buildings or civil engineering structural elements, but also smaller structures such as frames of trucks, automobiles or machines. Other mechanical systems contain beam structures and are designed and analyzed in a similar fashion. The transversal deflection of cables of suspension bridges can be modelled as that of a long beam with flexural rigidity varying with space under strong tension, with the bending moment at each point proportional to the local curvature. The question of localizing defects in a beam has been studied by many authors in the case of a beam without tension [8–11]. For example, Friswell [12] have compared the influence of different models of cracks in beam structures on the variation of the eigenfrequencies. To compare his modelizations with experimental data, he used the measures obtained in [13], in which a crack location can be found from the measured amplitudes at two points of the structure vibrating at one of its natural modes, the respective vibration frequency and an analytical solution of the dynamic response. Many authors use a sensitivity method based on the eigenvalue equation but most of them with a discrete form, i.e. with mass and stiffness matrices [14, 15]. In [14], Adhiakri proposed a method based on rates of change of eigenvalues and eigenvectors of a damped linear discrete dynamic system with respect to the system parameters. Results show the efficiency of the proposed method and the importance of including the changes of the mode shapes through a frequency-based damage detection algorithm. To localize and evaluate the change of the bending stiffness, Morassi [16] has also proposed a method based on generalized Fourier coefficients to calculate Eigenfrequency sensitivity to the stiffness variation caused by the damage.

In the articles cited above, the modification of mass is not taken into account. As tension is of interest for cables, in this article, we study the transverse deflection of an Euler-Bernoulli beam with tension. We establish a first order general estimate of the relative variation of each frequencies as a function of the simultaneous variations of different mechanical parameters of the beam such as tension, local density and local bending stiffness. Based on this estimation, a method to detect, localize and quantify a damage is proposed using only, at least, three frequencies of the undamaged and damaged beam, and the corresponding mode shapes (for a mass variation) and curvature mode shapes (for a stiffness variation) of the undamaged beam.

1. Modal parameters variations due to mechanical characteristics modifications

Several assumptions are made: the beam material is homogeneous, isotropic, and linearly elastic, the tension (axial force) is constant (positive or null) along the beam and viscous damping is ignored. The corresponding partial differential equation governing the transverse deflection $w(x,t))$ of an Euler-Bernoulli beam under axial force is:

$$\mu(x)\frac{\partial^2 w(x,t)}{\partial t^2} - T\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) = p(x,t) \tag{1}$$

where $w(x,t)$ is the transverse deflection at point $M(x)$ and time t, $\mu(x)$ the mass per unit length, T the tension force in the beam, $EI(x)$ the flexural rigidity (or bending stiffness) and $p(x,t)$ the excitation force per unit length. The particular case of the vibrating string is retrieved for $EI = 0$ (not studied here) and the case of the Euler-Bernoulli beam without tension is obtained for $T = 0$.

The normal modes ϕ_n are functions which are square integrable on $[0,L]$, as well as ϕ_n' and ϕ_n'' (i.e. $\phi_n \in H^2(0,L)$). By using the method of separation of variables, it can be shown that the normal modes ϕ_n of the beam verify the following equation:

$$(EI(x)\phi_n''(x))'' - T\phi_n''(x) - \lambda_n\mu(x)\phi_n(x) = 0 \tag{2}$$

for some boundary conditions. Multiplying this equation by any function $u(x)$ belonging to $H^2(0,L)$, an integration by parts leads to the relation:

$$\int_0^L \left(EI(x)\phi_n''(x)u''(x) + T\phi_n'(x)u'(x) - \lambda_n\mu(x)\phi_n(x)u(x)\right)dx + C = 0 \tag{3}$$

with C depending on the boundary conditions:

$$C = [\frac{\partial}{\partial x}(EI(x)\phi_n''(x))u(x)]_0^L - [EI(x)\phi_n''(x)u'(x)]_0^L - [T\phi_n'(x)u(x)]_0^L \tag{4}$$

If the beam is clamped in $x = a$ then $\phi_n(a) = \phi_n'(a) = 0$, if it is simply supported then $\phi_n(a) = \phi_n''(a) = 0$ and at a free end then $\phi_n''(a) = \phi_n^3(a) = 0$ (in such case the tension is equal to zero). For a beam verifying boundary conditions of the type listed above, the quantity C is equal to zero for any function $u(x)$ belonging to the space $H^2(0,L)$ and verifying the same boundary conditions as the normal modes.

Hence for a beam with classical boundary conditions, the following relation is verified for any function $u \in H^2(0,L)$ verifying the boundary conditions:

$$\int_0^L \left(EI(x)\phi_n''(x)u''(x) + T\phi_n'(x)u'(x) - \lambda_n\mu(x)\phi_n(x)u(x)\right)dx = 0 \tag{5}$$

We now consider changes of intrinsic characteristics of the beam due to local reduction or increase of flexural rigidity and weight in some area combined or not with a global variation of the tension. The modifications of the flexural rigidity and/or of the mass density will be qualified in the following by internal and that of the tension by external, respectively. We study the variation of the modal parameters due to these modifications which are assumed to be small in order to perform first order estimates. The modified mass and flexural rigidity are denoted by $\widetilde{EI}(x)$ and $\widetilde{\mu}(x)$ with:

$$\begin{aligned} \widetilde{EI}(x) &= EI(x) + \Delta EI(x) \\ \widetilde{\mu}(x) &= \mu(x) + \Delta\mu(x) \end{aligned} \tag{6}$$

where the variation of the density and of the bending stiffness are denoted by $\Delta\mu(x)$ and $\Delta EI(x)$ respectively. The tension is assumed to be constant along the beam. The variation of the tension is denoted by ΔT and the modified tension is equal to:

$$\widetilde{T} = T + \Delta T. \tag{7}$$

These changes induce a variation of the mode shapes and of the eigenvalues:

$$\begin{aligned} \tilde{\phi}_n(x) &= \phi_n(x) + \Delta\phi_n(x) \\ \tilde{\lambda}_n &= \lambda_n + \Delta\lambda_n \end{aligned} \tag{8}$$

For any function $u \in H^2(0,L)$ verifying the boundary conditions, the relation (5) must be verified by the new normal modes $\tilde{\phi}_n$:

$$\int_0^L \left(\widetilde{EI}(x)\widetilde{\phi}_n''(x)u''(x) + \widetilde{T}\widetilde{\phi}_n'(x)u'(x) - \tilde{\lambda}_n\widetilde{\mu}(x)\tilde{\phi}_n(x)u(x) \right) dx = 0 \tag{9}$$

As we want to perform an estimation to the first order, the following approximations are made:

$$\tilde{\lambda}_n\ \tilde{\mu}_n\tilde{\phi}_n - \lambda_n\ \mu_n\phi_n \approx \lambda_n\ \mu_n\ \Delta\phi_n + (\Delta\lambda_n\ \mu_n + \lambda_n\Delta\mu_n)\phi_n$$

The same type of approximation is made for T :$\widetilde{T}\ \tilde{\phi}_n - T\ \phi_n = \tilde{T}\ \Delta\phi_n + \Delta T\ \phi_n \approx T\ \Delta\phi_n + \Delta T\ \phi_n$, and the same is done for EI.

The difference of (9) minus (5) is zero for any $u \in H^2(0,L)$ verifying the boundary conditions. The variation $\Delta\phi_n$ can be expressed in the basis of normal modes $\{\phi_k\}_{k=1}^{\infty}$, as:

$$\Delta\phi_n(x) = \sum_{k=1}^{\infty} \alpha_{nk}\phi_k(x) \tag{10}$$

This yields the following first order estimate of the variation of the eigenvalue:

$$\Delta\lambda_n \int_0^L \mu(x)\,\phi_n^2(x)dx \approx \int_0^L \Delta EI(x)\phi_n''^2(x)dx + \Delta T \int_0^L \phi_n'^2(x)dx - \lambda_n \int_0^L \Delta\mu(x)\,\phi_n^2(x)dx \tag{11}$$

This equation can be modified so as to express the eigenvalue relative variation:

$$\frac{\Delta\lambda_n}{\lambda_n} \approx \frac{\gamma_n}{1+\gamma_n} \frac{\int_0^L \Delta EI(x)\,(\phi_n''(x))^2\,dx}{\int_0^L EI(x)\,(\phi_n''(x))^2\,dx} + \frac{\Delta T}{T}\frac{1}{(1+\gamma_n)} - \frac{\int_0^L \Delta\mu(x)\phi_n^2(x)dx}{\int_0^L \mu(x)\,\phi_n^2(x)dx} \tag{12}$$

For the beam without tension, it reads:

$$T = 0, \qquad \frac{\Delta\lambda_n}{\lambda_n} \approx \frac{\int_0^L \Delta EI(x)\,(\phi_n''(x))^2\,dx}{\int_0^L EI(x)\,(\phi_n''(x))^2\,dx} - \frac{\int_0^L \Delta\mu(x)\phi_n^2(x)dx}{\int_0^L \mu(x)\,\phi_n^2(x)dx} \tag{13}$$

The external modifications (mass and/or bending stiffness) are assumed to be localized around x_0 on an interval Ω of length ΔL with ΔL very small with respect to L, the total length of the beam. Moreover, since λ_n varies as the square of the frequency f_n then $2\frac{\Delta f_n}{f_n} = \frac{\Delta\lambda_n}{\lambda_n}$. In the simple case of rectangular functions and if the bending stiffness and the density are assumed to be initially constant ($EI(x) = EI$ and $\mu(x) = \mu$), it leads to:

$$2\frac{\Delta f_n}{f_n} \approx \frac{\gamma_n}{1+\gamma_n}\frac{\Delta EI\,\Delta L}{EI}\frac{(\phi_n''(x_0))^2}{\|\phi_n''\|_2^2} + \frac{\Delta T}{T}\frac{1}{(1+\gamma_n)} - \frac{\Delta\mu\,\Delta L}{\mu}\frac{\phi_n^2(x_0)}{\|\phi_n\|_2^2} \tag{14}$$

This relation is established for each mode. It permits to link the ratio of the variation of the frequency f_n to the ratio of the variation of the bending stiffness, of the mass density and of the tension as soon as the corresponding initial mode shape and mode curvature are known. It can also be rewritten as:

$$\frac{\Delta f_n}{f_n} \approx \frac{1}{2(1+\gamma_n)}\frac{\Delta T}{T} + G_n(x_0)\frac{\Delta EI\Delta L}{EIL} - H_n(x_0)\frac{\Delta\mu\Delta L}{\mu L} \tag{15}$$

$$\text{with} \qquad G_n(x) = \frac{\gamma_n}{2(1+\gamma_n)}\frac{L\phi_n''^2(x)}{\|\phi_n''\|_2^2}, \qquad H_n(x) = \frac{1}{2}\frac{L\phi_n^2(x)}{\|\phi_n\|_2^2} \tag{16}$$

2. LOCALIZATION PROCEDURE

The objective of this part is to use the relation (15) concerning the variation of each frequency to localize the modifications. We assume that we know the p first frequencies of two states (an initial state and a modified one) and that the corresponding mode shapes of the initial state are known. The modifications of the mass and of the bending stiffness are assumed to be localized around the abscissa x_0 on a small interval of length ΔL (with $\Delta L \ll L$) and the tension is assumed to be measurable in the two states. For simplicity the modification is assumed constant on the interval $[x_0 - \frac{\Delta L}{2}, x_0 + \frac{\Delta L}{2}]$, in this case the relation (15) leads to:

$$\Delta_n(x_0) \approx G_n(x_0)\frac{\Delta EI\Delta L}{EIL} - H_n(x_0)\frac{\Delta\mu\Delta L}{\mu L} \tag{17}$$

$$\text{with} \qquad \Delta_n(x_0) = \frac{\Delta f_n}{f_n} - \frac{1}{2(1+\gamma_n)}\frac{\Delta T}{T} \tag{18}$$

If the frequencies and the tension are measured and as γ_n depends on the characteristics of the beam and can be evaluated, then Δ_n is a known quantity. The functions $G_n(x)$, $H_n(x)$ depending of the mode shapes of the initially state are assumed to be known too. The terms $\frac{\Delta EI\Delta L}{EIL}$ and $\frac{\Delta\mu\Delta L}{\mu L}$ are unknown and represent the relative variation of the bending stiffness and of the mass.

To localize the modification, the idea is to use (17) for three different frequencies and to take into account the fact that these three equations with two unknowns are compatible, which means that the determinant of order 3 formed by the coefficients Δ, G and H is equal to zero for $x = x_0$. So if we define the function $F_{m,n,k}(x)$ by:

$$\begin{aligned} F_{m,n,k}(x) = {} & G_m(x)H_n(x)\Delta_k(x) + G_n(x)H_k(x)\Delta_m(x) + G_k(x)H_m(x)\Delta_n(x) + \\ & - G_n(x)H_m(x)\Delta_k(x) - G_k(x)H_n(x)\Delta_m(x) - G_m(x)H_k(x)\Delta_n(x) \end{aligned} \tag{19}$$

this function should be null for $x = x_0$. The value x_0 is a minimizer of $|F_{m,n,k}(x)|$ for any triplet (m,n,k). So, for the states before and after the modification, if the three first eigenvalues are measured and the corresponding mode shapes are known, then it is possible to evaluate x_0 as $x_m = ArgminF_{1,2,3}(x)$. Besides, if eigenvalues of higher order are known, it is possible to improve the identification of the possible values of x_0 by taking into account these complementary informations. So if the p first eigenvalues (with $p \geq 3$) are available, the possible values of x_0 can be obtained as the values of x that minimize the quantity $Q_p(x)$ defined below:

$$Q_p(x) = \sum_{1 \leq n \leq m \leq k \leq p} |F_{m,n,k}(x)| \tag{20}$$

If the frequencies are not distinct, the function $F_{m,n,k}(x)$ is equal to zero, so only the case of distinct frequencies is relevant.

It should be noted that the quantity $Q_p(x)$ increases with the number p of used frequencies for each fixed value of x. It also can be observed that the function Q_p increases globally with the importance of the variation of the mechanical parameters. The maximum of the function Q_p is linked to the amplitude of the modification. It will be interesting to establish a relation between $\|Q_p\|_\infty$ and a measure of the modification.

3. Application to beams without tension

The experimental tests consist of a free-free beam that have been described and used in [16]. The specimen was suspended from above by means of two soft springs, so as to simulate free-free boundary conditions.

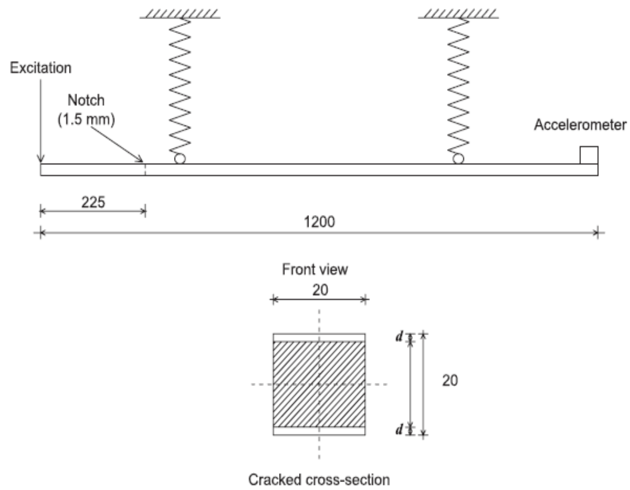


Figure 1 : Experimental model of bending vibrating beam and damage configuration (Lengths in *mm*)

It should be recalled that the free-free beam has a double multiplicity zero eigenvalue, corresponding to two independent rigid body motions. These vibrating modes are insensitive to damage and will be omitted in the sequel. The damage consisted of two symmetric notchs placed at the cross-section at 0.255 m from the left end (see Figure 1). Their depth was progressively increased by $1mm$ at a time from the undamaged configuration to a final level of damage corresponding to a depth of $6mm$ on both sides of the cross-section. For each level, the lowest seven natural frequencies were measured according to an impulse technique (Table 1).

In this case the only unknown quantity is the relative variation of the bending stiffness so:

$$\frac{\Delta f_n}{f_n} \approx G_n(x_0) \frac{\Delta EI \Delta L}{EIL} = \frac{L \phi_n''^2(x)}{\|\phi_n''\|_2^2} \frac{\Delta EI \Delta L}{EIL} \tag{21}$$

Table 1 : Experimental frequencies of a beam with notches of varying depth

Frequency	**Undamaged**	**1** *mm*	**2** *mm*	**3** *mm*	**4** *mm*	**5** *mm*	**6** *mm*
f_1	72.19	72.19	72.16	72.16	72.06	71.94	71.53
f_2	198.40	198.31	198.06	197.68	196.44	194.69	189.25
f_3	387.73	387.50	386.84	385.33	381.56	374.97	360.96
f_4	639.72	639.38	638.41	636.28	630.81	623.78	607.03
f_5	951.47	951.31	950.75	950.03	947.03	943.47	935.16
f_6	1320.56	1320.56	1320.34	1320.25	1319.97	1319.97	1319.16
f_7	1747.03	1746.81	1746.63	1746.13	1742.88	1739.41	1728.28

The use of two different frequencies is sufficient to eliminate the unknown. This case with a single unknown has been addressed in [17,18] and a similar relation has been used in slightly different cases (rod with a crack or a beam with a crack represented by the insertion of a massless rotational spring at the damaged cross section). In our case, the measure obtained for other low frequencies are also taken into account in the function $Q_p(x)$ which is defined by:

$$Q_p(x) = \sum_{1 \leq n \leq m \leq p} |F_{m,n}(x)| \tag{22}$$

with in the case of a single modification

$$F_{m,n}(x) = \frac{\Delta f_n}{f_n} G_m(x) - \frac{\Delta f_m}{f_m} G_n(x) \tag{23}$$

As in the general case, an estimate of the localization should be obtained by $x_m = Argmin \left[\widehat{Q}_p(x) \right]$ and an indication of the magnitude should be given by the maximum of the function $Q_p(x)$.

In [16], the aim was to localize the modification by estimating the value of the bending stiffness along the beam using Fourier type series. The partial development uses from 2 to 7 frequencies. The notch is supposed to correspond to the minimal value of the bending stiffness. The results obtain for notches of 3 *mm* depth are shown in Figure 2 and the localization is obtained approximatively for $p = 3, 4$ and 5 but completely fails for $p = 2, 6, 7$. As the number of used frequencies increases, a lot of perturbations appears in the estimation of the bending stiffness. Moreover this method does not succeed to localize small defects of 1 or 2 *mm*.

We now apply our method to this beam with a 3 *mm* notch. In Figure 3, the graphes of Q_p and of the normed function $\widehat{Q}_p$, are represented for $p = 3, 4, 5, 6, 7$. Due to the symmetry of the initial problem, the square of the initial mode shape and mode curvature are even functions with respect to the mid-point and so two symmetric minima can be observed.

The precision of the localization of x_m is quite good for less than five eigenfrequencies and it increases with p as it can be seen on the graph of the normed function.

CONCLUSION

This paper discusses the development of a procedure for the localization of modifications from the knowledge of the transverse vibrations of an Euler-Bernoulli beam with or without tension. The first natural frequencies, the corresponding mode shapes and curvature mode shapes of the initial state, and only the first natural frequencies of the modified state are assumed to be known.

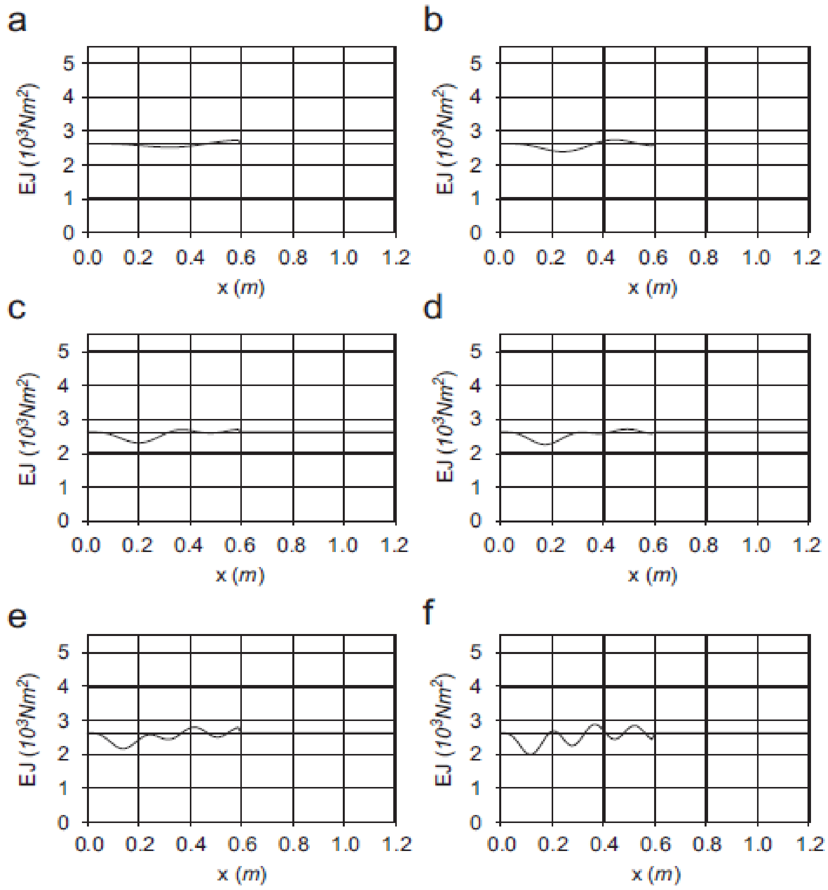
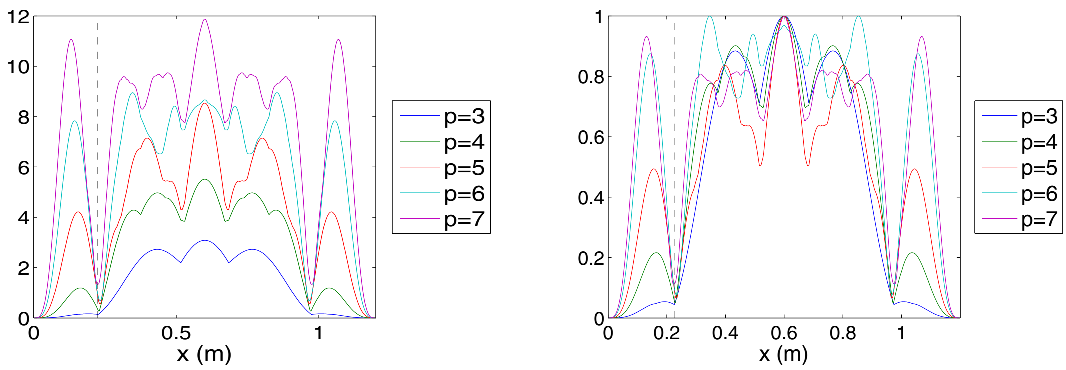


Figure 2 : Estimated bending stiffness EI for $3mm$ with p frequencies used and damage location at $x_0 = 0.225m$ (a) p=2, (b) $p = 3$, (c) $p = 4$, (d) $p = 5$, (e) $p = 6$ and (f) $p = 7$.



(a) $Q_p(x)$ (b) $\widehat{Q}_p(x)$

Figure 3 : Free-free beam with a notch of $3mm$ depth at $x_0 = 0.225m$

Expanding the variation of the eigen vectors of the modified state on the basis of the eigen vectors of the unmodified state, a first order relation is first established in the general case, for each mode, between the relative variation of the natural frequency, of the tension, of the bending stiffness and of the mass density.

When internal modifications (of bending stiffness and/or mass density) are localized around a point x_0, these analytical estimations of the variations of the natural frequencies as function of the external modification, allowed us to build a function of the position x along the beam, which is theoretically null when $x = x_0$. This function can be normed so that its value varies between 0 and 1.

A procedure of localization based on the searching of the global minimum of this function was performed and it has proved to provide a good estimate of x_0.

REFERENCES

[1] D. Broda, W.J. Staszewski, A. Martowicz, T. Uhl, and V.V. Silberschmidt. Modelling of nonlinear crack-wave interactions for damage detection based on ultrasounda review. *Journal of Sound and Vibration*, 333(4):1097–1118, 2014.

[2] J.-H. Chou and Ghaboussi J. Genetic algorithm in structural damage detection. *Computers & Structures*, 79(14):1335–1353, 2014.

[3] O.S. Salawu. Detection of structural damage through changes in frequency: a review. *Engineering Structures*, 19(9):718–723, 1997.

[4] S.W. Doebling, C.R. Farrar, and M.B. Prime. A summary review of vibration-based damage identification methods. *Shock and Vibration Digest*, 30:91–105, 1998.

[5] M. Gobbato, J.P. Conte, J.B. Kosmatka, and C.R. Farrar. A reliability-based framework for fatigue damage prognosis of composite aircraft structures. *Probabilistic Engineering Mechanics*, 29:176–188, 2012.

[6] Y. Lei, J. Lin, Z. He, and M.J. Zuo. A review on empirical mode decomposition in fault diagnosis of rotating machinery. *Mechanical Systems and Signal Processing*, 35(1-2):108–126, 2013.

[7] G. Sposito, C. Ward, P. Cawley, P.B. Nagy, and Scruby C. A review of non-destructive techniques for the detection of creep damage in power plant steels. *NDT & E International*, 43(7):555–567, 2010.

[8] A.K. Pandey, M. Biswas, and M.M. Samman. Damage detection from changes in curvature mode shapes. *Journal of Sound and Vibration*, 145:321–332, 1991.

[9] A.K. Pandey and M. Biswas. Damage detection in structures using changes in flexibility. *Journal of Sound and Vibration*, 169:3–17, 1994.

[10] M. Cao, L. Ye, L. Zhou, Z. Su, and Bai R. Sensitivity of fundamental mode shape and static deflection for damage identification in cantilever beams. *Mechanical Systems and Signal Processing*, 25(2):630–643, 2011.

[11] Z.A. Jassim, N.N. Ali, F. Mustapha, and Abdul Jalil N.A. A review on the vibration analysis for a damage occurrence of a cantilever beam. *Engineering Failure Analysis*, 31:442–461, 2013.

[12] M.I. Friswell and J.E.T. Penny. Crack modeling for structural health monitoring. *Structural Health Monitoring*, 1(2):139–148, 2002.

[13] P.F. Rizos, N. Aspragathos, and A.D. Dimarogonas. Identification of crack location and magnitude in a cantilever beam from the vibration modes. *Journal of Sound and Vibration*, 138(3):381–388, 1990.

[14] S. Adhiakri. Rates of change of eigenvalues and eigenvectors in damped dynamic system. *AIAA Journal*, 39(11):1452–1457, 1999.

[15] A. Esfandiari, F. Bakhtiari-Nejad, and A. Rahai. Theoretical and experimental structural damage diagnosis method using natural frequencies through an improved sensitivity equation. *International Journal of Mechanical Sciences*, 70:79–89, 2013.

[16] A. Morassi. Damage detection and generalized Fourier coefficients. *Journal of Sound and Vibration*, 302(1-2):229–259, 2007.

[17] A. Morassi. Identification of a crack in a rod based on changes in a pair of natural frequencies. *Journal of Sound and Vibration*, 242(4):577–596, 2001.

[18] M. Dilena and A. Morassi. The use of antiresonances for crack detection in beams. *Journal of Sound and Vibration*, 276(1-2):195–214, 2004.